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CERTAIN RESULTS ON q-STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS

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Abstract: In the present paper, we introduce two Subclasses $S_s^*(\delta, \theta, q)$ and $S_c^*(\delta, \theta, q)$ of analytic and q-starlike functions with respect to symmetric and conjugate points in the open unit disk Δ. For functions belonging to these Classes, we obtain estimates coefficient and sufficient Condition.

Keywords and Phrases: q-Starlike with respect to symmetric points, q-derivative, univalent function.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions k of the form

$$k(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j. \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{\xi : |\xi| < 1\}$.

Quantum Calculus or q-Calculus is the study of Calculus without limits and in recent years, it has attracted attention of many researchers due to its vast applications in Mathematics and Physics. At the beginning of the last century studies on q-difference equations appeared in intensive works especially by Jackson, [5, 6], Kac, V. and Cheung, P. [7], Noor, K. [9], Ezeafulukwe, U. A., and Darus, M. [3], and Ucar, H. [12], as well as numerous other contributors in the field. Research work in connection with function theory and q-theory together was first introduced by Ismail et al. [4]. Till now only non-significant interest in this area was shown although it deserves more attention. Jackson [5, 6], introduced and studied q-derivative and q-integral in a systematic way. We would like to mention that the use of q-derivative (or q-difference operator D_q) was introduced in [5].

The differential operator $D_q k(\xi)$, acting on $k \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, is defined by

$$D_q k(\xi) = \frac{k(\xi) - k(q\xi)}{(1-q)\xi}, \quad \xi \neq 0, \quad q \neq 1, \tag{1.2}$$

where

$$D_q k(0) = k'(0) \quad \textit{and} \quad D_q^2 k(\xi) = D_q(D_q k(\xi)) \quad .$$

From (1.2), we can deduce that

$$D_q k(\xi) = 1 + \sum_{j=2}^{\infty} [j]_q c_j \xi^{j-1}, \tag{1.3}$$

where

$$[j]_q = \frac{1-q^j}{1-q}. \tag{1.4}$$

As $q \to 1^-$, $[j]_q \to j$.

Jackson [6], introduced the q-integral:

$$\int_0^{\xi} k(t) d_q(t) = (1-q)\xi \sum_{j=0}^{\infty} q^j k(q^j \xi), \tag{1.5}$$

provided that the series converges.

Let S_s^* be the subclass of S consisting of functions given by (1.1) satisfying $\Re\left\{\dfrac{\xi k'(\xi)}{k(\xi)-k(-\xi)}\right\} > 0$ for all $\xi \in \Delta$. These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [10]. Recently, ELAshwa and Thomas [2], have introduced various results concerning functions in S_s^* and two other classes namely the class S_c^* of starlike functions with respect to conjugate points and the class S_{cs}^* of starlike functions with respect to symmetric conjugate points.

In this work, using the concept of q-derivative with symmetric and conjugate points we define the following.

Definition 1.1. *A function $k \in \mathcal{A}$, is said to be in the class $S_s^*(\delta,\theta,q)$, if the following condition is hold*

$$\left|\frac{\xi D_q k(\xi)}{k(\xi)-k(-\xi)}-1\right| < \theta\left|\frac{\delta\xi D_q k(\xi)}{k(\xi)-k(-\xi)}+1\right|, \xi \in \Delta,$$

where $0 \le \delta \le 1$, $0 < \theta \le 1, 0 < q \le 1$.

Definition 1.2. *A function $k \in \mathcal{A}$, is said to be in the class $S_c^*(\delta,\theta,q)$, if the following condition is hold*

$$\left|\frac{\xi D_q k(\xi)}{k(\xi)+\overline{k(\overline{\xi})}}-1\right| < \theta\left|\frac{\delta\xi D_q k(\xi)}{k(\xi)+\overline{k(\overline{\xi})}}+1\right|, \xi \in \Delta,$$

where $0 \le \delta \le 1$, $0 < \theta \le 1, 0 < q \le 1$.

The classes $S_s^*(\delta,\theta,q)$ and $S_c^*(\delta,\theta,q)$ are yield several known subclasses of $\mathcal{A}$, namely $S_s^*(\delta,\theta,1) = S_s^*(\delta,\theta)$ and $S_c^*(\delta,\theta,1) = S_c^*(\delta,\theta)$ are introduced and studied by Sudharsan, T. et al. [11], and as $\delta = \theta = 1$ and $q \to 1^-$ we get the classes S_s^* and S_c^* which introduced by El-Ashwa, R. et al. [2].

2. Main Results

We need a lemma of Lakshminarasimhan [8].

Lemma 2.1. *Let $h(\xi)$ be analytic in Δ and satisfy the condition*

$$\left|\frac{1-h(\xi)}{1+\delta h(\xi)}\right| < \theta, \tag{2.1}$$

$\xi \in \Delta$, $0 \le \delta \le 1$, $0 < \theta \le 1$, *with,* $h(0) = 1$. *Then we have*

$$h(\xi) = \frac{1-\xi\upsilon(\xi)}{1+\delta\xi\upsilon(\xi)}, \tag{2.2}$$

where $\upsilon(\xi)$ is analytic in Δ and $|\upsilon(\xi)| \leq \theta$ for $\xi \in \Delta$. Conversely any function $h(\xi)$ given by (2.2) above is analytic in Δ, and satisfies (2.1).

We now prove a lemma, this is used to obtain the coefficient estimates for functions in the class $S_s^*(\delta, \theta, q)$ and $S_c^*(\delta, \theta, q)$.

Lemma 2.2. *Let* $k(\xi)$ *and* $P(\xi)$ *belong to* $\mathcal{S}$ *and satisfy the condition*

$$\left|\frac{\xi D_q k(\xi)}{p(\xi)} - 1\right| < \theta \left|\frac{\delta \xi D_q k(\xi)}{p(\xi)} + 1\right|, \tag{2.3}$$

$0 \leq \delta \leq 1$, $0 < \theta \leq 1$, $0 < q \leq 1$ *and* $\xi \in \Delta$, *with* $k(\xi)$ *given by (1.1), and* $p(\xi) = \xi + \sum_{j=2}^{\infty} d_j \xi^j$. *Then for* $j \geq 2$

$$|d_j - [j]_q c_j|^2 \leq 2(\delta\theta^2 + 1) \sum_{t=1}^{j-1} [t]_q |c_t||d_t|, (|c_1| = |d_1| = [1]_q = 1)\,. \tag{2.4}$$

Proof. By lemma 2.1, we have

$$\frac{\xi D_q k(\xi)}{p(\xi)} = \frac{1 - \xi \upsilon(\xi)}{1 + \delta \xi \upsilon(\xi)},$$

$\upsilon(\xi)$ is analytic in Δ and $|\upsilon(\xi)| < \theta$ for $\xi \in \Delta$. Then

$$\xi D_q k(\xi)[1 + \delta \xi \upsilon(\xi)] = p(\xi)[1 - \xi \upsilon(\xi)].$$

Or equivalently

$$[\delta \xi D_q k(\xi) + p(\xi)]\xi \upsilon(\xi) = p(\xi) - \xi D_q k(\xi).$$

Now if,

$$\psi(\xi) = \xi \upsilon(\xi) = \sum_{j=1}^{\infty} r_j \xi^j,$$

then

$$|\psi(\xi)| \leq \theta |\xi| \ \ for \ \ \xi \in \Delta.$$

Therefore

$$\left(\delta(\xi + \sum_{j=2}^{\infty} [j]_q c_j \xi^j) + \xi + \sum_{j=2}^{\infty} d_j \xi^j\right) \left(\sum_{j=1}^{\infty} r_j \xi^j\right) = \xi + \sum_{j=2}^{\infty} d_j \xi^j - \left(\xi + \sum_{j=2}^{\infty} [j]_q c_j \xi^j\right).$$

Such as

$$\left(\delta\xi+\xi+\sum_{j=2}^{\infty}(\delta[j]_q c_j+d_j)\xi^j\right)\left(\sum_{j=1}^{\infty}r_j\xi^j\right)=\sum_{j=2}^{\infty}d_j\xi^j-\sum_{j=2}^{\infty}[j]_q c_j\xi^j.$$

Thus,

$$\left((\delta+1)\xi+\sum_{j=2}^{\infty}(\theta[j]_q c_j+d_j)\xi^j\right)\left(\sum_{j=1}^{\infty}r_j\xi^j\right)=\sum_{j=2}^{\infty}\left(d_j-[j]_q c_j\right)\xi^j. \tag{2.5}$$

Equating the coefficient of ξ^j in (2.5), we have

$$c_j-[j]_q c_j=(\delta+1)r_{j-1}+(\delta[2]_q c_2+d_2)r_{j-2}+...+(\delta[j-1]_q c_{j-1}+d_{j-1})r_1. \tag{2.6}$$

Thus the coefficient combination on the right side of (2.6) depends only upon the coefficients combination $(\delta[2]_q c_2+d_2)$, $(\delta[3]_q c_3+d_3)$, . . . ,$(\delta[j-1]_q c_{j-1}+d_{j-1})$ on the left side. Hence for $j\geq 2$ we can write

$$\left((\delta+1)\xi+\sum_{t=2}^{j-1}(\delta[t]_q c_t+d_t)\xi^t\right)\psi(\xi)=\sum_{t=2}^{j}\left(d_t-[t]_q c_t\right)\xi^t+\sum_{t=j+1}^{\infty}e_t\xi^t. \tag{2.7}$$

Squaring the moduli of both sides of (2.7) and integrating along $|\xi|=\lambda<1$ and on using the fact that $|\psi(\xi)|\leq\theta|\xi|$, we obtain

$$\sum_{t=2}^{j}|d_t-[t]_q c_t|^2\lambda^{2t}+\sum_{t=j+1}^{\infty}|e_t|^2\lambda^{2t}\leq\theta^2\left((\delta+1)^2\lambda^2+\sum_{t=2}^{j-1}|\delta[t]_q c_t+d_t|^2\lambda^{2t}\right).$$

Letting $\lambda\to 1$ on the left side of this inequality, we obtain

$$\sum_{t=2}^{j}|d_t-[t]_q c_t|^2\leq\theta^2\left((\delta+1)^2+\sum_{t=2}^{j-1}|\delta[t]_q c_t+d_t|^2\right).$$

We can rewrite as

$$\begin{aligned}|d_j-[j]_q c_j|^2&<\theta^2(\delta+1)^2+\theta^2\sum_{t=2}^{j-1}|\delta[t]_q c_t+d_t|^2-\sum_{t=2}^{j-1}|d_t-[t]_q c_t|^2\\&<\theta^2(\delta+1)^2+\theta^2\sum_{t=2}^{j-1}\left(|\delta[t]_q c_t|^2+2\delta[t]_q c_t d_t+|d_t|^2\right)\\&-\sum_{t=2}^{j-1}\left(|d_t|^2-2|[t]_q c_t d_t|+[t]_q^2|c_t|^2\right).\end{aligned}$$

This implies that

$$|d_j - [j]_q c_j|^2 \leq \theta^2(\delta+1)^2 + (\theta^2\delta^2 - 1)\sum_{t=2}^{j-1}[t]_q^2|c_t|^2$$
$$+(\theta^2-1)\sum_{t=2}^{j-1}|d_t|^2 + (2\delta\theta^2+2)\sum_{t=2}^{j-1}[t]_q|c_t d_t|. \tag{2.8}$$

Thus,

$$\begin{aligned}|d_j - [j]_q c_j|^2 &\leq (2\delta\theta^2+2)\left(1+\sum_{t=2}^{j-1}[t]_q|c_t||d_t|\right)\\ &\leq (2\delta\theta^2+2)\sum_{t=1}^{j-1}[t]_q|c_t||d_t|, \ \ (|c_1| = |d_1| = [1]_q = 1).\end{aligned}$$

Or equivalent

$$|d_j - [j]_q c_j|^2 \leq 2\delta\theta^2\sum_{t=1}^{j-1}[t]_q|c_t||d_t| + 2\sum_{t=1}^{j-1}[t]_q|c_t||d_t|.$$

Theorem 2.3. *Let k and p belong to $\mathcal{S}$ and be given as in Lemma 2.2. Then for $j \geq 2$*

$$|d_j - [j]_q c_j|^2 \leq 2(\delta\theta^2+1)CA_q\left(1-\frac{1}{[j]_q}, k\right)^{\frac{1}{2}} A_q\left(1-\frac{1}{[j]_q}, p\right)^{\frac{1}{2}},$$

where $A_q(\lambda, k)$ denotes the area enclosed by $k(|\xi| = \lambda)$ and where C is a constant.

Proof. We have by (2.4) of lemma 2.2

$$|d_j - [j]_q c_j|^2 \leq 2(\delta\theta^2+1)\sum_{t=1}^{j-1}[t]_q|c_j||d_j|, (|c_1| = |d_1| = [1]_q = 1).$$

The Cauchy-Schwarz inequality gives for $0 < \lambda < 1$

$$\begin{aligned}|d_j - [j]_q c_j|^2 &\leq 2\delta\theta^2\left(\sum_{t=1}^{j-1}[t]_q|c_t|^2\right)^{\frac{1}{2}}\left(\sum_{t=1}^{j-1}[t]_q|d_t|^2\right)^{\frac{1}{2}}\\ &\quad + 2\left(\sum_{t=1}^{j-1}[t]_q|c_t|^2\right)^{\frac{1}{2}}\left(\sum_{t=1}^{j-1}[t]_q|d_t|^2\right)^{\frac{1}{2}}\\ &\leq \frac{2\delta\theta^2}{r^{2j}}\left(\sum_{t=1}^{j-1}[t]_q|c_t|^2\lambda^{2t}\right)^{\frac{1}{2}}\left(\sum_{t=1}^{j-1}[t]_q|d_t|^2\lambda^{2t}\right)^{\frac{1}{2}}\end{aligned}$$

$$+\frac{2}{\lambda^{2j}}\left(\sum_{t=1}^{j-1}[t]_q|c_t|^2\lambda^{2t}\right)^{\frac{1}{2}}\left(\sum_{t=1}^{j-1}[t]_q|d_t|^2\lambda^{2t}\right)^{\frac{1}{2}}$$
$$\leq \frac{2\delta\theta^2}{\pi\lambda^{2j}}A_q\left(\lambda,k\right)^{\frac{1}{2}}A_q\left(\lambda,p\right)^{\frac{1}{2}}+\frac{2}{\pi\lambda^{2j}}A_q(\lambda,k)^{\frac{1}{2}}A_q(\lambda,p)^{\frac{1}{2}},$$

since $A_q(\lambda,k)=\pi\sum_{j-1}^{\infty}[j]_q|c_j|^2\lambda^{2j}$.
Choosing $\lambda = 1-\frac{1}{[j]_q}$ for $j\geq 2$, the result follows.

Theorem 2.4. *Let $k\in S_s^*(\delta,\theta,q)$ and be given by (1.1). Then*

i) $([j]_{q^2})^2|c_{2j}|^2\leq \frac{2(\delta\theta^2+1)}{1+2q+q^2}\sum_{l=1}^{j}[2l-1]_q|c_{2l-1}|^2,\ \ j\geq 1, (|c_1|=[1]_q=1)$.

ii) $(1-[j]_{q^2})^2|c_{2j-1}|^2\leq \frac{2q^2(\delta\theta^2+1)}{1+2q+q^2}\sum_{l=1}^{j-1}[2l-1]_q|c_{2l-1}|^2,\ \ j\geq 2$.
Further, if $\delta\theta<1$,

iii) $([j]_{q^2})^2|c_{2j}|^2\leq \frac{(\theta^2-1)}{1+2q+q^2}\sum_{l=1}^{j}|c_{2l-1}|^2+\frac{2(\theta+1)}{1+2q+q^2}\sum_{l=1}^{j}[2l-1]_q|c_{2l-1}|^2,\ \ j\geq$
1,
$(|c_1|=[1]_q=1)$ *and*

iv) $(1-[j]_{q^2})^2|c_{2j-1}|^2\leq \frac{q^2(\theta^2-1)}{1+2q+q^2}\sum_{l=1}^{j-1}|c_{2l-1}|^2+\frac{2q^2(\theta+1)}{1+2q+q^2}\sum_{l=1}^{j-1}[2l-1]_q|c_{2l-1}|^2,$
$j\geq 2$.

The inequalities (i) and (ii) are sharp.

Proof. Since $k\in S_s^*(\delta,\theta,q)$, by Lemma 2.1 we have $\xi\frac{D_qk(\xi)}{p(\xi)}=h(\xi)$, where p is an odd starlike function with $p(\xi)=\frac{k(\xi)-k(-\xi)}{2}$ and $h(\xi)=\frac{1-\xi\upsilon(\xi)}{1+\delta\xi\upsilon(\xi)}$, where $\upsilon(\xi)$ is analytic in Δ, and $|\upsilon(\xi)|<\theta$ for $\xi\in\Delta$. We have,

$$\xi\frac{D_qk(\xi)}{p(\xi)}=\frac{1-\xi\upsilon(\xi)}{1+\delta\xi\upsilon(\xi)},$$

then

$$\xi\frac{2D_qk(\xi)}{k(\xi)-k(-\xi)}=\frac{1-\xi\upsilon(\xi)}{1+\delta\xi\upsilon(\xi)},$$

We can write as

$$[2\delta\xi D_q k(\xi) + k(\xi) - k(-\xi)]\xi \upsilon(\xi) = k(\xi) - k(-\xi) - 2\xi D_q k(\xi).$$

Now, if

$$\psi(\xi) = \xi\upsilon(\xi) = \sum_{j=1}^{\infty} r_j \xi^j,$$

then

$$|\psi(\xi)| = |\xi\upsilon(\xi)| \leq \theta|\xi| \ \ for \ \ \xi \in \Delta.$$

Therefore

$$[(2\delta+2)\xi + \sum_{j=2}^{\infty}[2\delta[j]_q + (1-(-1)^j)]c_j\xi^j] \sum_{j=1}^{\infty} r_j\xi^j = \sum_{j=2}^{\infty}(1-(-1)^j - 2[j]_q)c_j\xi^j. \quad (2.9)$$

Hence for $j \geq 2$ we can write

$$[(2\delta+2)\xi + \sum_{j=2}^{\infty}[2\delta[j]_q + (1-(-1)^j)]c_j\xi^j] \sum_{j=1}^{\infty} r_j\xi^j = \sum_{t=2}^{j}[1-(-1)^t - 2[t]_q]c_t\xi^t + \sum_{t=j+1}^{\infty} e_t\xi^t. \quad (2.10)$$

Squaring the moduli of both sides of (2.10) and integrating along $|\xi| = \lambda < 1$ and on using the fact that $|\psi(\xi)| \leq \theta|\xi|$, we obtain

$$\sum_{t=2}^{j}(1-(-1)^t - 2[t]_q)^2|c_t|^2\lambda^{2t} + \sum_{t=j+1}^{\infty}|e_t|^2\lambda^{2t} < \theta^2\{(2\delta+2)^2\lambda^2 + \sum_{t=2}^{j-1}[2\delta[t]_q + (1-(-1)^t)]^2|c_t|^2\lambda^{2t}\}.$$

Letting $\lambda \to 1$ on the left side of this inequality, we obtain

$$\sum_{t=2}^{j}(1-(-1)^t - 2[t]_q)^2|c_t|^2 < \theta^2\left[(2\delta+2)^2 + \sum_{t=2}^{j-1}(2\delta[t]_q + (1-(-1)^t))^2|c_t|^2\right].$$

This implies that

$$|1-(-1)^j - 2[j]_q|^2|c_j|^2 \leq \theta^2\left[(2\delta+2)^2 + \sum_{t=2}^{j-1}(2\delta[t]_q + (1-(-1)^t))^2|c_t|^2\right] - \sum_{t=2}^{j-1}(1-(-1)^t - 2[t]_q)^2|c_t|^2.$$

Hence

$$\begin{aligned}|1-(-1)^j-2[j]_q|^2|c_j|^2 &\leq 4\theta^2(\delta+1)^2+4(\delta^2\theta^2-1)\sum_{t=2}^{j-1}[t]_q^2|c_t|^2\\ &+(\theta^2-1)\sum_{t=2}^{j-1}(1-(-1)^t)^2|c_t|^2\\ &+4(\delta\theta^2+1)\sum_{t=2}^{n-1}(1-(-1)^t)[t]_q|c_t|^2 \qquad (2.11)\\ &\leq 4\theta^2(\delta+1)^2+4(\delta\theta^2+1)\sum_{t=2}^{j-1}(1-(-1)^t)[t]_q|c_t|^2\\ &\leq 8(\delta\theta^2+1)\sum_{l=1}^{j}[2l-1]_q|c_{2l-1}|^2,(|c_1|=[1]_q=1).\end{aligned}$$

Therefore,

$$([j]_{q^2})^2|c_{2j}|^2 \leq \frac{2(\delta\theta^2+1)}{1+2q+q^2}\sum_{l=1}^{j}[2l-1]_q|c_{2l-1}|^2,(|c_1|=[1]_q=1). \qquad (2.12)$$

$$(1-[j]_{q^2})^2|c_{2j-1}|^2 \leq \frac{2q^2(\delta\theta^2+1)}{1+2q+q^2}\sum_{l=1}^{j-1}[2l-1]_q|c_{2l-1}|^2, j\geq 2. \qquad (2.13)$$

From (2.12) and (2.13) the inequalities (i) and (ii) follow.

Now if, $\delta\theta<1$ using (2.11), we get

$$\begin{aligned}|1-(-1)^j-2[j]_q|^2|c_j|^2 &\leq 4\theta^2(\delta+1)^2+4(\delta^2\theta^2-1)\sum_{t=2}^{j-1}[t]_q^2|c_t|^2\\ &+4(\delta\theta^2+1)\sum_{t=2}^{j-1}(1-(-1)^t)[t]_q|c_t|^2\\ &+(\theta^2-1)\sum_{t=2}^{j-1}(1-(-1)^t)^2|c_t|^2\end{aligned}$$

Thus,

$$([j]_{q^2})^2|c_{2j}|^2 \leq \frac{(\theta^2-1)}{1+2q+q^2}\sum_{l=1}^{j}|c_{2l-1}|^2+\frac{2(\theta+1)}{1+2q+q^2}\sum_{l=1}^{j}[2l-1]_q|c_{2l-1}|^2, j\geq 1. \qquad (2.14)$$

And

$$(1-[j]_{q^2})^2|c_{2j-1}|^2 \leq \frac{q^2(\theta^2-1)}{1+2q+q^2}\sum_{l=1}^{j-1}|c_{2l-1}|^2 + \frac{2q^2(\theta+1)}{1+2q+q^2}\sum_{l=1}^{j-1}[2l-1]_q|c_{2l-1}|^2, j \geq 2. \tag{2.15}$$

From (2.14) and (2.15) the inequalities (iii) and (iv) follow.

We observe, that the inequalities (i) and (ii) are sharp as can be seen from the function $k(\xi) = \frac{\xi}{2(\delta\theta^2+1)(1-q\xi)}$, but the inequalities (iii) and (iv) involve additional sum terms and are not necessarily sharp.
when $\delta = \theta = 1$ and $q \to 1^-$, we get the corresponding results of EL-Ashwah and Thomas [2].

Theorem 2.5. *If $k \in S_s^*(\delta, \theta, q)$ with $\delta\theta < 1$, then $c_j = 0\left(\frac{1}{[j]_q}\right)$ as $j \to \infty$.*

Proof. We observe that when $\delta\theta < 1$, for $k \in S_s^*(\delta, \theta, q)$, $\frac{\xi D_q k(\xi)}{k(\xi)-k(-\xi)}$ is bounded. We first prove that

$$[[j]_q - (1-(-1)^j)]^2|c_j|^2 \leq 4(\theta+1)\sum_{t=1}^{j-1}[t]_q|c_j|^2, (|c_1| = [1]_q = 1).$$

If $k \in S_s^*(\delta, \theta, q)$ is given by (1.1), we have using Lemma 2.1

$$\frac{\xi D_q k(\xi)}{k(\xi)-k(-\xi)} = \frac{1-\xi\upsilon(\xi)}{1+\delta\xi\upsilon(\xi)},$$

where $\upsilon(\xi)$ is analytic in Δ, and $|\upsilon(\xi)| < \theta$ for $\xi \in \Delta$. Then

$$\xi D_q k(\xi)[1+\delta\xi\upsilon(\xi)] = (k(\xi)-k(-\xi))[1-\xi\upsilon(\xi)].$$

Or equivalently

$$[\delta\xi D_q k(\xi) + k(\xi) - k(-\xi)]\xi\upsilon(\xi) = k(\xi) - k(-\xi) - \xi D_q k(\xi).$$

Now if,

$$\psi(\xi) = \xi\upsilon(\xi) = \sum_{j=1}^{\infty} r_j \xi^j,$$

then

$$|\psi(\xi)| = |\xi\upsilon(\xi)| \leq \theta|\xi| \;\; for \;\; \xi \in \Delta.$$

Therefore

$$\left[(\delta+2)\xi+\sum_{j=2}^{\infty}\left(\delta[j]_q+(1-(-1)^j)\right)c_j\xi^j\right]\sum_{j=1}^{\infty}r_j\xi^j=\left[\xi+\sum_{j=2}^{\infty}\left(1-(-1)^j-[j]_q\right)c_j\xi^j\right]. \tag{2.16}$$

Equating the coefficient of ξ^j in (2.16), we have
$(1-(-1)^j-[j]_q)=(\delta+2)r_{j-1}+(\delta[2]_qc_2+(1-(-1)^2))r_{j-2}+...+(\delta[j-1]_qc_{j-1}+(1-(-1)^{j-1}))r_1$.

Thus the coefficient combination on the right side of (2.16) depends only upon the coefficients combination
$(\delta[2]_qc_2+(1-(-1)^2))$, $(\delta[3]_qc_3+(1-(-1)^3))$, . . . ,$(\delta[j-1]_qc_{j-1}+(1-(-1)^{j-1}))$ on the left side. Hence for $j\geq 2$ we can write

$$\left[(\delta+2)\xi+\sum_{t=2}^{j-1}\left(\delta[t]_q+(1-(-1)^t)c_t\xi^t\right)\right]\psi(\xi)=\sum_{t=2}^{j}(1-(-1)^t-[t]_q)c_t\xi^t+\sum_{t=j+1}^{\infty}e_t\xi^t. \tag{2.17}$$

Squaring the moduli of both sides of (2.17) and integrating along $|\xi|=\lambda<1$ and on using the fact that $|\psi(\xi)|\leq\theta|\xi|$, we obtain

$$\sum_{t=2}^{j}([t]_q-1+(-1)^t)^2|c_t|^2\lambda^{2t}+\sum_{t=j+1}^{\infty}|e_t|^2\lambda^{2t}<\theta^2\{(\delta+2)^2\lambda^2 \\ +\sum_{t=2}^{j-1}(\delta[t]_q+(1-(-1)^j))^2|c_t|^2\lambda^{2t}\}.$$

Letting $\lambda\to 1$ on the left side of this inequality, we obtain

$$\sum_{t=2}^{j}([t]_q-1+(-1)^t)^2|c_t|^2<\theta^2\left[(\delta+2)^2+\sum_{t=2}^{j-1}(\delta[t]_q+(1-(-1)^t))^2|c_t|^2\right].$$

This implies that

$$\begin{aligned}([j]_q-(1-(-1)^j))^2|c_j|^2&<\theta^2(\delta+2)^2+\theta^2\sum_{t=2}^{j-1}(\delta[t]_q+(1-(-1)^t))^2|c_t|^2\\&\quad-\sum_{t=2}^{j-1}([t]_q-(1-(-1)^t))^2|c_t|^2\\&\leq\theta^2(\delta+2)^2+(\theta^2\delta^2-1)\sum_{t=2}^{j-1}[t]_q^2|c_t|^2+(\theta^2-1)\sum_{t=2}^{j-1}(1-(-1)^t)^2|c_t|^2\end{aligned}$$

$$+ (2\delta\theta^2 + 2)\sum_{t=2}^{j-1}(1-(-1)^t)[t]_q|c_t|^2$$

Or equivalently

$$\begin{aligned}([j]_q - (1-(-1)^j))^2|c_j|^2 \leq & 4\theta\sum_{t=1}^{j-1}[t]_q|c_t|^2 + 4\sum_{t=1}^{j-1}[t]_q|c_t|^2 \\ & \leq 4(\theta+1)\sum_{t=1}^{j-1}[t]_q|c_t|^2, (|c_1| = [1]_q = 1). \qquad (2.18)\end{aligned}$$

Since $\delta\theta < 1$. It remains to show that $c_j = 0\left(\frac{1}{[j]_q}\right)$ as $j \to \infty$. From (2.18) we have

$$([j]_q - (1-(-1)^j))^2|c_j|^2 \leq 4(\theta+1)\left(1 + \sum_{t=2}^{j-1}[t]_q|c_t|^2\right). \qquad (2.19)$$

Since $\dfrac{\xi D_q k(\xi)}{k(\xi) - k(-\xi)}$ is bounded, it follows that $k(\xi)$ is bounded. Now following Clunie and Keogh [1], we conclude that Σ, the area of the image of $k(\xi)$ is given by

$$\Sigma = \pi\left(1 + \sum_{t=2}^{\infty}[t]_q|c_t|^2\right), \qquad (2.20)$$

and consequently, $\sum_{t=2}^{\infty}[t]_q|c_t|^2 < \infty$ and hence $\lambda_j = \sum_{t=2}^{\infty}[t]_q|c_t|^2 \to 0$ as $j \to \infty$. Thus we have

$$\sum_{t=2}^{j-1}[t]_q|c_t|^2 = \sum_{t=2}^{j-1}(\lambda_t - \lambda_{t+1}) = \lambda_2 - \lambda_j = 0(1) \;\; as \;\; j \to \infty. \qquad (2.21)$$

Using (2.19) and (2.21), we have $c_j = 0(\frac{1}{[j]_q})$ as $j \to \infty$.

Theorem 2.6. *Let* $k(\xi) = \xi + \sum_{j=2}^{\infty} c_j\xi^j$*, be analytic in the unit disc* Δ*. If for* $\xi \in \Delta$*,* $0 \leq \delta \leq 1$*,* $0 < \theta \leq 1$*,* $0 < q \leq 1$

$$\sum_{j=2}^{\infty}\left[\frac{(1+\theta\delta)[j]_q}{\theta(2+\delta)-1} + \frac{(\theta-1)(1-(-1)^j)}{\theta(2+\delta)-1}\right]|c_j| \leq 1,$$

or equivalently,

$$\sum_{t=1}^{\infty}\left[\frac{(1+\theta\delta)([2]_q[t]_{q^2})|c_{2t}|}{\theta(2+\delta)-1}-\frac{(1+\delta\theta)(1+q[2]_q[t]_{q^2})|c_{2t+1}|+2(\theta-1)|c_{2t+1}|}{\theta(2+\delta)-1}\right]\leq 1, \tag{2.22}$$

then $k(\xi)$ belongs to the class $S_s^*(\delta,\theta,q)$.

Proof. We use the method of Clunie and Keogh [1]. Suppose that $k(\xi)=\xi+\sum_{j=2}^{\infty}c_j\xi^j$, then for $|\xi|<1$

$$\begin{aligned}
&|\xi D_qk(\xi)-k(\xi)+k(-\xi)|-\theta|\delta\xi D_qk(\xi)+k(\xi)-k(-\xi)|\\
&=\left|\xi+\sum_{j=2}^{\infty}[j]_qc_j\xi^j-2\xi-\sum_{j=2}^{\infty}(1-(-1)^j)c_j\xi^2\right|-\theta\left|\delta\xi+\delta\sum_{j=2}^{\infty}[j]_qc_j\xi^j+2\xi+\sum_{j=2}^{\infty}(1-(-1)^j)c_j\xi^2\right|\\
&=\left|-\xi+\sum_{j=2}^{\infty}\left([j]_q-1+(-1)^j\right)c_j\xi^2\right|-\theta\left|(\delta+2)\xi+\sum_{j=2}^{\infty}\left(\delta[j]_q+1-(-1)^j\right)c_j\xi^2\right|\\
&\leq|-\xi|+\sum_{j=2}^{\infty}\left([j]_q-1+(-1)^j\right)|c_j||\xi|^2-\theta(\delta+2)|\xi|+\theta\sum_{j=2}^{\infty}\left[\delta[j]_q+1-(-1)^j\right]|c_j||\xi|^2\\
&\leq\lambda+\sum_{j=2}^{\infty}\left([j]_q-1+(-1)^j\right)|c_j|\lambda^2-\theta(\delta+2)\lambda+\theta\sum_{j=2}^{\infty}\left[\delta[j]_q+1-(-1)^j\right]|c_j|\lambda^2,\ \ |\xi|=\lambda<1\\
&\leq\left[\sum_{j=2}^{\infty}\left[(1+\delta\theta)[j]_q-1+(-1)^j+\theta(1-(-1)^j)\right]|c_j|-\theta(\delta+2)+1\right]\lambda\\
&\leq\left[\sum_{t=1}^{\infty}(1+\delta\theta)[2t]_q|c_{2t}|+\sum_{t=1}^{\infty}[(1+\delta\theta)[2t+1]_q|c_{2t+1}|+2(\theta-1)|c_{2t+1}|]-\theta(\delta+2)+1\right]\lambda\\
&\leq\left[\sum_{t=1}^{\infty}(1+\delta\theta)([2]_q[t]_{q^2})|c_{2t}|+\sum_{t=1}^{\infty}\left[(1+\delta\theta)(1+q[2]_q[t]_{q^2})|c_{2t+1}|+2(\theta-1)|c_{2t+1}|\right]-\theta(\delta+2)+1\right]\lambda\\
&\leq 0\ \ by\ \ (2.22).
\end{aligned}$$

Therefore it follows that for $|\xi|<1$

$$\left|\left(\frac{\xi D_qk(\xi)}{k(\xi)-k(-\xi)}-1\right)/\left(\frac{\delta\xi D_qk(\xi)}{k(\xi)-k(-\xi)}+1\right)\right|<\theta,$$

so that $k(\xi)\in S_s^*(\delta,\theta,q)$.

We note that

$$k(\xi)=\xi-\frac{\theta(\delta+2)-1}{(1+\delta\theta)[j]_q+(\theta-1)(1-(-1)^j)}\xi^j$$

is an extremal function with respect to the theorem, since

$$\left|\left(\frac{\xi D_q k(\xi)}{k(\xi)-k(-\xi)}-1\right)/\left(\frac{\delta\xi D_q k(\xi)}{k(\xi)-k(-\xi)}+1\right)\right| = \theta,$$

for $\xi = 1,\ 0 \leq \delta \leq 1, \frac{1}{2} < \theta \leq 1, 0 < q \leq 1,\ j = 2, 3, ...$

3. Illustrative Example

To illustrate the practical application of the sufficient condition established in Theorem 2.6, we consider a specific case by assigning particular values to the parameters δ, θ and q.

Example. Let $\delta = 1, \theta = 0.6$ and $q = 0.5$. Substituting these values into the inequality 2.22, the condition for a function $k(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j$ to belong to the class $S_s^*(1, 0.6, 0.5)$ becomes:

$$\sum_{j=2}^{\infty}\left[\frac{(1+(0.6))[j]_{\frac{1}{2}}}{0.6(2+1)-1}+\frac{(0.6-1)(1-(-1)^j)}{0.6(2+1)-1}\right]|c_j| \leq 1,$$

Simplifying the coefficients:

For even terms ($j = 2t$), we have $\frac{1.6[2t]_{\frac{1}{2}}}{0.8} = 2[2t]_{\frac{1}{2}}$

For odd terms ($j = 2t + 1$), we have $\frac{1.6[2t+1]_{\frac{1}{2}}}{0.8} = 2[2t + 1]_{\frac{1}{2}}$

Thus, the function $k(\xi)$ belongs to $S_s^*(1, 0.6, 0.5)$ if:

$$\sum_{t=1}^{\infty} 2[2t]_{\frac{1}{2}}|c_{2t}| + \sum_{t=1}^{\infty}(2[2t+1]_{\frac{1}{2}})|c_{2t+1}| \leq 1,$$

Now, for even terms ($j = 2$):

The first even coefficient c_2, where $[2]_{\frac{1}{2}} = 1 + 0.5 = 1.5$, so the first part of the term is 3, and the second part is 0, therefore the coefficient for $|c_2|$ is 3.

For odd terms ($j = 3$):

The first odd coefficient c_3, where $[3]_{\frac{1}{2}} = 1+0.5+0.25 = 1.75$, so the first part of the term is 3.5, and the second part is -1, therefore the coefficient for $|c_3| = 3.5-1 = 2.5$.

Hence, the function $k(\xi) \in S_s^*(1, 0.6, 0.5)$ if: $3|c_2| + 2.5|c_3| + ... \leq 1$

Now, if we define a simple analytic function $k_0(\xi) = \xi + 0.2\xi^2 + 0.1\xi^3$, we can verify the condition: $(3 \times 0.2) + (2.5 \times 0.1) = 0.6 + 0.25 = 0.85 \leq 1$, and this confirmed that $k_0(\xi) \in S_s^*(1, 0.6, 0.5)$.

Theorem 3.1. *Let $k(\xi) \in S_c^*(\delta, \theta, q)$ and be given by (1.1). Then for $j \geq 2$*

$$([j]_q + 1)^2 |c_q|^2 \leq 2(\delta\theta^2 + 1) \left(\sum_{t=1}^{j} [t]_q |c_t|^2 \right). \tag{3.1}$$

Proof. The theorem follows immediately from Lemma 2.2.

The inequality in the above Theorem is sharp, and a function that demonstrates sharpness is

$$k(\xi) = \frac{\xi}{2(1+\delta\theta^2)(1-q)} \left(\frac{1}{1-\xi} - \frac{q}{1-q\xi} \right).$$

4. Conclusion

In this work, using the concept of q-derivative with symmetric and conjugate points we define, two subclasses, namely $S_s^*(\delta, \theta, q)$ and $S_c^*(\delta, \theta, q)$ of analytic and q-starlike functions with respect to symmetric and conjugate points in the open unit disk Δ. We derived coefficient estimates and established the necessary sufficient condition for functions belong to $S_s^*(\delta, \theta, q)$. Furthermore, we provided detailed coefficient estimates the class $S_c^*(\delta, \theta, q)$, contributing to a deeper understanding of their geometric properties.

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